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LETTER TO THE EDITOR

Real forms of $U_q(OSp(1|2))$ and quantum $D=2$ supersymmetry algebras

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Abstract. We present three real forms of quantum superalgebra $U_q(OSp(1|2))$. By defining suitable contraction limits we describe the q-deformations of $D=2$ superPoincaré and $D=2$ superEuclidean algebras as Hopf bialgebras.

The aim of this letter is to consider the real forms of quantum superalgebra $U_q(OSp(1|2))$ and perform different contraction limits, providing respectively quantum $D=2$ Euclidean and quantum $D=2$ Minkowski superalgebras. We perform these limits for the whole Hopf bialgebra structure of the real form of $U_q(OSp(1|2))$, in order to obtain, after contraction, genuine quantum algebras [1, 2]. It appears that in some contraction limits we need to supplement the rescaling of the generators with the change of scale of the deformation parameter q, approaching $q=1$ in a way firstly proposed for $U_q(SU(2))$ by the Firenze group [3–5].

The quantum superalgebra $U_q(OSp(1|2))$ as well as its dual object, quantum group $OSP_q(1|2)$ were discussed extensively by Kulish [6–8], and the discussion of real forms of $U_q(Sp(2))$ can be found in [9]. By considering firstly the real forms of the conventional superalgebra $OSp(1|2))$ ($q=1$) we obtain three involutions†.

(i) Two equivalent ones, describing the superalgebra $OSp(1|2; R)$ with the non-compact bosonic sector $Sp(2; R) \simeq SU(1.1)$.

(ii) A third one, denoted in [11] by $UOSp(1|2)$, with compact bosonic sector $SU(2)$ and with the natural involution described by graded adjoint operation [10, 11]. Then we describe the extensions of these real forms to $q \neq 1$. It appears that similarly to the non-supersymmetric case of $U_q(Sp(2)) = U_q(sl(2))$ (see [9]) the degeneracy of real forms is removed, i.e. there are three real forms of $U_q(OSp(1|2))$ which are not equivalent. We then consider three contractions of the real form $U_q(OSp(1|2))$—two providing $D=2$ Minkowski quantum supersymmetry algebra and one providing $D=2$ Euclidean quantum supersymmetry algebra. We describe their complete Hopf algebra structure (multiplication, comultiplication, antipode) as well as their Casimirs, obtained by considering the contraction of q-deformed Casimir for $U_q(OSp(1|2))$. Finally we present comments and mention the relation with the q-oscillator realization of $U_q(OSp(1|2))$.

† For the general discussion of involutions defining real forms of superalgebras see [10].

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The conventional ($q=1$) OSp(1|2) superalgebra is defined by its Cartan–Chevaley basis ($e_\alpha, e_{-\alpha}, h_\alpha$) as follows:

$$\{e_\alpha, e_{-\alpha}\} = h_\alpha \qquad [h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha} \tag{1a}$$

where $e_\alpha, e_{-\alpha}$ are odd (fermionic) generators. The Cartan–Weyl basis describing all generators of OSp(1|2) is obtained by introducing the defining relations for the bosonic generators $e_{2\alpha}, e_{-2\alpha}$, corresponding to double roots:

$$\{e_{\pm\alpha}, e_{\pm\alpha}\} = e_{\pm 2\alpha}. \tag{1b}$$

The relations (1b) imply that

$$[e_{2\alpha}, e_{-2\alpha}] = -8h_\alpha \qquad [h_\alpha, e_{\pm 2\alpha}] = \pm 4e_{\pm 2\alpha} \tag{1c}$$

$$[e_{\pm 2\alpha}, e_{\mp\alpha}] = \pm 4e_{\pm\alpha} \qquad [e_{\pm 2\alpha}, e_{\pm\alpha}] = 0. \tag{1d}$$

The relations (1c) describe the bosonic subalgebra Sp(2)~SL(2). Comparing with standard formulae, the change of sign in the first formula (1c) should be observed. The physical O(2, 1; C) basis is given by the formulae

$$L_1 = -\tfrac{1}{8}(e_{2\alpha} + e_{-2\alpha}) \qquad L_2 = 2 - \tfrac{1}{8}(e_{2\alpha} - e_{-2\alpha}) \tag{2}$$

permitting us to write (1c) as follows

$$[L_1, L_2] = -L_3 \qquad [L_2, L_3] = L_1 \qquad [L_3, L_1] = -L_2 \tag{3}$$

i.e. describing the $D=3$ Lorentz group with the signature ($-++$) (L_1 compact, L_2, L_3 non-compact). Using the formulae

$$V_\pm = \frac{1}{2\sqrt{2}} e_{\mp\alpha} \tag{4}$$

one gets

$$\{V_+, V_+\} = \tfrac{1}{2}(L_2 - L_1) \qquad \{V_-, V_-\} = -\tfrac{1}{2}(L_1 + L_2) \qquad \{V_+, V_-\} = -\tfrac{1}{2}L_3 \tag{5a}$$

and

$$[L_1, V_\pm] = \pm\tfrac{1}{2}V_\mp \qquad [L_2, V_\pm] = \tfrac{1}{2}V_\mp \qquad [L_3, V_\pm] = \pm\tfrac{1}{2}V_\pm \tag{5b}$$

with the following Casimir:

$$C_2^{q=1} = -L_1^2 + L_2^2 + \left(L_3 - \frac{1}{4}\right)^2 + 2V_+V_-. \tag{6}$$

The real forms of the superalgebra generated by the set of bosonic generators B_i and fermionic generators F_r can be described [10, 11] by the invariance under the automorphisms, which can be represented in the following three equivalent ways:

(a) the conjugation

$$B_i \to \tau(B_i) \qquad F_r \to \tau(F_r) \qquad \tau(A \cdot A') = \tau(A) \cdot \tau(A') \qquad (A = B_i, F_r) \tag{7}$$

(b) the adjoint operation

$$B_i \to B_i^+ = -\tau(B_i) \qquad F_r \to F_r^+ = \pm \mathrm{i}\tau(F_r) \qquad (A \cdot A')^+ = (A')^+ A^+ \qquad (A = B_i, F_r) \tag{8}$$

(*c*) the graded adjoint operation

$$B_i \to B_i^{\#} = -\tau(B_i) \qquad F_r \to F_r^{\#} = \pm\tau(F_r) \qquad (A = B_i, F_r) \tag{9}$$
$$(A \cdot A')^{\#} = (-1)^{\mathrm{grad}A \cdot \mathrm{grad}A'}(A')^{\#}A^{\#}$$

where grad $B_i = 0$ and grad $F_r = 1$. Further, we shall consider only the involution (*b*) which seems to be well adjusted to the quantum mechanical realization of the generators.

For the superalgebra OSp(1|2) one can introduce the following three adjoint operations, leaving the relations (1*a*–*d*) invariant:

(i)

$$H_\alpha^+ = -H_\alpha \qquad e_{\pm 2\alpha}^+ = -e_{\pm 2\alpha} \qquad e_{\pm\alpha}^+ = \mathrm{i}e_{\pm\alpha} \tag{10a}$$

(ii)

$$\tilde{h}_\alpha^+ = \tilde{h}_\alpha \qquad \tilde{e}_{\pm 2\alpha}^+ = \tilde{e}_{\mp 2\alpha} \qquad \tilde{e}_{\pm\alpha}^+ = \tilde{e}_{\mp\alpha} \tag{10b}$$

(iii)

$$H_\alpha^+ = H_\alpha \qquad E_{\pm 2\alpha}^+ = -E_{\pm 2\alpha} \qquad E_{\pm\alpha} = \pm\mathrm{i}E_{\mp\alpha}. \tag{10c}$$

We shall assume that for the real form of the superalgebra the bosonic generators are anti-Hermitian ($B_i = -B_i^+$), and fermionic generators satisfy the relation $f_r = \mathrm{i}F_r^+$, i.e. the bilinear supersymmetry relations have real coefficients. From (10*a*–*c*) we get the following.

(i) For (10*a*) we get the formulae (2), (3) with $L_k^+ = -L_k$, ($k = 1, 2, 3$) and (5*a*, *b*) with $V_\pm^+ = \mathrm{i}V_\pm$. In such a way we obtain the superalgebra OSp(1|2; *R*).

(ii) For (10*b*) the generators $\tilde{L}_k^+ = -\tilde{L}_k$; we choose

$$\tilde{L}_1 = \frac{\mathrm{i}}{2}(\tilde{e}_{2\alpha} + \tilde{e}_{-2\alpha}) \qquad \tilde{L}_2 = \tfrac{1}{8}(\tilde{e}_{2\alpha} - \tilde{e}_{-2\alpha}) \tag{11}$$

which also satisfy the O(2, 1) algebra:

$$[\tilde{L}_1, \tilde{L}_2] = \tilde{L}_3 \qquad [\tilde{L}_2, \tilde{L}_3] = -\tilde{L}_1 \qquad [\tilde{L}_3, \tilde{L}_1] = -\tilde{L}_2. \tag{12}$$

Introducing $\tilde{V}_\pm = \frac{1}{4}(\tilde{e}_\alpha \mp \mathrm{i}\tilde{e}_{-\alpha})$ which satisfy $\tilde{V}_\pm^+ = \mathrm{i}\tilde{V}_\pm$ the remaining superalgebra relations are

$$\{\tilde{V}_+, \tilde{V}_+\} = \tfrac{1}{2}(\tilde{L}_3 - \tilde{L}_2) \qquad \{\tilde{V}_-, \tilde{V}_-\} = \tfrac{1}{2}(\tilde{L}_3 + \tilde{L}_2) \qquad \{\tilde{V}_+, \tilde{V}_-\} = \tfrac{1}{4}\tilde{L}_1 \tag{13a}$$

$$[\tilde{L}_1, \tilde{V}_\pm] = -\tfrac{1}{2}\tilde{V}_\pm \qquad [\tilde{L}_2, \tilde{V}_\pm] = -\tfrac{1}{2}\tilde{V}_\mp \qquad [\tilde{L}_3, \tilde{V}_\pm] = \mp\tfrac{1}{2}\tilde{V}_\mp. \tag{13b}$$

If we observe that the algebras (3) and (12) can be identified if $L_k = \tilde{L}_{k+2}$ (mod 3), it can be checked that the remaining relations (5*a*, *b*) and (13*a*, *b*) are the same if $\tilde{V}_\pm = (1/\sqrt{2})(V_+ \pm V_-)$. The involutions (10*a*) in its bosonic sector describe the real algebra Sp(2; *R*), and the involution (10*b*)—the real algebra SU(1, 1)†. Because Sp(2; *R*) ≃ SU(1, 1), their supersymmetric extensions by only fermionic generators also have to be the same.

(iii) For (10*c*) we can choose

$$M_1 = \tfrac{1}{8}(E_{2\alpha} + E_{-2\alpha}) \qquad M_2 = \frac{\mathrm{i}}{8}(E_{2\alpha} - E_{-2\alpha}) \qquad M_3 = \frac{\mathrm{i}}{4}H_\alpha \tag{14}$$

† We would like to point out that due to the presence of the 'minus' sign on the RHS of the first of the formulae (1*c*) these involutions are *not* the same as in [9].

satisfying $M_k^+ = -M_k$ and the O(3) algebra

$$[M_i, M_j] = \varepsilon_{ijk} M_k. \tag{15}$$

The supersymmetry algebra relations can be obtained below (see equation (29)) by putting $q=1$. The choice of the '+' operation (10*c*) leads to the real superalgebra with compact bosonic sector SU(2) which has been discussed in detail in [11] and denoted by UOSp(1|2). It should be stressed that the relations (10*c*) imply that in the fermionic sector $(F^+)^+ = -F$, i.e. it cannot be represented by a conventional Hermitian congregation of complex matrices. In such a case it appears useful to use as the automorphism the graded adjoint operation (9).

The relations (1*a*–*d*) are q-deformed in the following way:

$$\{e_\alpha, e_{-\alpha}\} = [h_\alpha]_q \qquad [h_\alpha, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha} \tag{16a}$$

$$\tfrac{1}{2}(1+q^{\pm 2})\{e_{\pm\alpha}, e_{\pm\alpha}\} = e_{\pm 2\alpha} \tag{16b}$$

$$[e_{2\alpha}, e_{-2\alpha}] = (1+q^2)(1+q^{-2})\{1-[2h_\alpha+1]_q + q^{-2}(1-q)^2[h_\alpha]_q e_\alpha e_{-\alpha}\} \tag{16c}$$

$$[h_\alpha, e_{\pm 2\alpha}] = \pm 4 e_{\pm 2\alpha} \tag{16d}$$

$$[e_{\mp 2\alpha}, e_{\pm\alpha}] = \pm(1+q^{\mp 2})(q^{h_\alpha+1} + q^{-(h_\alpha+1)}) e_{\mp\alpha} \tag{16e}$$

$$[e_{\pm 2\alpha}, e_{\pm\alpha}] = 0. \tag{16f}$$

Introducing the 'physical' generators L_i (see (2)) and extending the formulae (4) for $q \neq 1$ as follows

$$V_\pm = \tfrac{1}{2}(q+q^{-1})^{-1/2} e_{\mp\alpha} \tag{17}$$

one can rewrite the relations (16*c*–*e*) as the deformed O(2, 1) algebra

$$\begin{aligned} &[L_1, L_2] = -\tfrac{1}{32} q^{-2}(1+q^2)\{1-(q+q^{-1})[\tfrac{1}{4} - 4L_3]_{q^2} - 4(q^4-1)^2[2L_3]_{q^2} V_- V_+\} \\ &[L_2, L_3] = L_1 \qquad [L_3, L_1] = -L_2. \end{aligned} \tag{18}$$

The first commutator shows that $U_q(\mathrm{Sp}(2)) \sim U_q(\mathrm{O}(2,1))$ is not a quantum subgroup of $U_q(\mathrm{OSp}(1|2))$.

The fermionic sector of the q-deformed Cartan–Weyl basis of OSp(1|2) takes the form:

$$\begin{aligned} &\{V_+, V_+\} = \frac{2q}{(q+q^{-1})^2}(L_2 - L_1) \\ &\{V_-, V_-\} = -\frac{2q^{-1}}{(q+q^{-1})^2}(L_1 + L_2) \\ &\{V_+, V_-\} = -\tfrac{1}{4}[2L_3]_{q^2} \end{aligned} \tag{19}$$

and

$$\begin{aligned} &[L_1, V_\pm] = \pm\tfrac{1}{8}(1+q^{\pm 2})(q^{4L_3 \pm 1} + q^{-4L_3 \mp 1}) V_\mp \\ &[L_2, V_\pm] = \tfrac{1}{8}(1+q^{\pm 2})(q^{4L_3 \pm 1} + q^{-4L_3 \mp 1}) V_\mp \\ &[L_3, V_\pm] = \pm\tfrac{1}{2} V_\pm. \end{aligned} \tag{20}$$

From the coproduct relations in the Cartan–Chevaley basis

$$\Delta(e_{\pm\alpha}) = e_{\pm\alpha} \otimes q^{h_\alpha/2} + q^{-h_\alpha/2} \otimes e_{\pm\alpha} \qquad \Delta(h_\alpha) = h_\alpha \otimes 1 + 1 \otimes h_\alpha \tag{21}$$

and the q-deformed defining relations (16b), one obtains the following coproduct formulae

$$\begin{aligned}\Delta(L_1)&=L_1\otimes q^{-4L_3}+q^{4L_3}\otimes L_1-\tfrac{1}{2}(q+q^{-1})q^{2L_3}\{(q^4-1)V_-\otimes V_-\\&\quad+(q^{-4}-1)V_+\otimes V_+\}q^{-2L_3}\\\Delta(L_2)&=L_2\otimes q^{-4L_3}+q^{4L_3}\otimes L_2+\tfrac{1}{2}(q+q^{-1})q^{2L_3}\{(q^{-4}-1)V_+\otimes V_+\\&\quad-(q^4-1)V_-\otimes V_-\}q^{-2L_3}\\\Delta(L_3)&=L_3\otimes 1+1\otimes L_3\\\Delta(V_\pm)&=V_\pm q^{-2L_3}+q^{2L_3}\otimes V_\pm.\end{aligned}\tag{22}$$

Supplementing the relations defining co-units

$$\varepsilon(L_k)=\varepsilon(V_\pm)=0\tag{23}$$

and antipodes:

$$\begin{aligned}S(L_1)&=-\tfrac{1}{2}(q^2+q^{-2})L_1+\tfrac{1}{2}(q^{-2}-q^2)L_2\\S(L_2)&=\tfrac{1}{2}(q^{-2}-q^2)L_1-\tfrac{1}{2}(q^2+q^{-2})l_2\\S(L_3)&=-L_3\\S(V_\pm)&=-q^{\mp1}V_\pm\end{aligned}\tag{24}$$

we obtain the q-deformation of the Cartan–Weyl basis for OSp(1|2) as a bialgebra satisfying the axioms of quantum group [1, 2].

Finally we observe that the quantum Casimir C_2^q takes the form ($q=e^\eta$)

$$C_2^q=\tfrac{1}{16}([1-4L_3]_q^2-1)-(L_1+L_2)(L_1-L_2)+2\cosh^2\eta\cosh\eta(2-4L_3)V_+V_-\tag{25}$$

where we have chosen the constants in a way providing the standard limit (6) for $q=1$. For the quantum superalgebra U_q(OSp(1|2)) one can introduce the invariance under the involutions (10a–c) provided that q is restricted in a suitable way ($q=q^*$ or $|q|=1$). Let us introduce two types of automorphisms of the coproducts $\Delta(a)=b_i\otimes c_i$ under the involutions $a\to a^+$.

(i) Coalgebra automorphism:

$$\Delta(a^+)=(\Delta(a))^+=\sum_i b_i^+\otimes c_i^+\tag{26}$$

(ii) Anti-coalgebra automorphism:

$$\Delta(a^+)=(\Delta'(a))^+=\sum_i c_i^+\otimes b_i^+\tag{27}$$

where $\Delta'=\tau\Delta=\Sigma_i c_i\otimes b_i$ (τ-flip automorphism).

One can write down the following table.

Table 1. The real forms of the complex quantum superalgebra U_q(OSp(1|2)), corresponding to the involutions (10a–c).

Involution	Automorphism of superalgebra	Automorphism of coalgebra	Automorphism of anti-coalgebra	Name of real form
(10a)	$q=q^*$	$\|q\|=1$	$q=q^*$	U_q(OSp(1|2; R))
(10b)	$\|q\|=1$	$q=q^*$	$\|q\|=1$	$\tilde{U}_q$(OSp(1|2; R))
(10c)	$\|q\|=1$	$q=q^*$	$\|q\|=1$	U_q(UOSp(1|2))

One can show that the automorphism of superalgebra relations implies the same restrictions on q as the automorphism of the table of antipodes.

From table 1 it follows that

(i) The quantum deformation of the real superalgebra OSp(1|2; R) leads to *different* real quantum superalgebras for the involutions (10a) and (10b), i.e. the deformation removes the degeneracy, described above. Similarly, as in the case of isomorphic algebras Sp(2; R) $\simeq$ SU(1, 1) where for U_q(Sp(2)) we have $|q|=1$, one obtains two different real quantum algebras, with q taking respectively values on the unit circle ($\tilde{U}_q$(OSp(1|2; R)) or q real (U_q(OSp(1|2; R)). In both cases the '+' involution generating real structure is the automorphism of the anticoalgebra.

(ii) The third involution leads to the quantum superalgebra U_q(UOSp(1|2)) where $|q|=1$. It is described by three anti-Hermitian bosonic generators M_i (see (14)) and two complex supercharges $S_\pm = E_{\pm\alpha}$, where

$$(S_\pm)^+ = \pm \mathrm{i} S_\mp \tag{28}$$

as follows:

$$\begin{aligned} &[M_1, M_2] = -\frac{\mathrm{i}}{32} q^{-2}(1+q^2)\{1-(q+q^{-1})[\tfrac{1}{2}-4\mathrm{i}M_3]_{q^2} - q(q^2-1)[2\mathrm{i}m_3]_{q^2} S_- S_+\} \\ &\qquad [M_2, M_3] = M_1 \qquad [M_3, M_1] = M_2 \end{aligned} \tag{29a}$$

and the basic supersymmetry relations are:

$$\begin{aligned} \{S_+, S_+\} &= \frac{8}{1+q^2}(M_1 - \mathrm{i}M_2) \\ \{S_-, S_-\} &= \frac{8}{1+q^{-2}}(M_1 + \mathrm{i}M_2) \\ \{S_+, S_-\} &= -[4\mathrm{i}M_3]_q. \end{aligned} \tag{29b}$$

(*a*) *Quantum D=2 Euclidean* SUSY. It is clear from the relation (3) that $L_1 = T$ describes the compact subgroup O(2) of O(2, 1) algebra. We propose therefore the following rescaling of the superalgebra (18–20)

$$L_2 = RP_1 \qquad L_3 = 3RP_2 \qquad L_1 = T \qquad V_\pm = \sqrt{R}\, Q_\pm \tag{30}$$

and simultaneous rescaling of real deformation parameter q (see [3–5, 12]; κ-mass-like parameter)

$$q(R) = \exp\left(\frac{1}{2\kappa R}\right). \tag{31}$$

We obtain by putting $T \to \infty$

$$\begin{aligned} &\{Q_\pm, Q_\pm\} = \pm\tfrac{1}{2}P_1 \qquad \{Q_+, Q_-\} = -\frac{\kappa}{4}\sinh\frac{2P_2}{\kappa} \\ &[T, Q_\pm] = \pm\tfrac{1}{2}\cosh\frac{2P_2}{\kappa}\, Q_\mp \qquad [P_1, Q_\pm] = \tfrac{1}{2}\cosh\frac{2P_2}{\kappa}\, Q_\mp \qquad [P_2, Q_\pm] = 0 \\ &[P_1, P_2] = 0 \qquad [T, P_1] = -\tfrac{1}{4}\kappa\sinh\frac{4P_2}{\kappa} \qquad [T, P_2] = P_1. \end{aligned} \tag{32}$$

The Casimir (25) before the contraction limit $R \to \infty$ should be rescaled as follows

$$C_2^{q(R)} = R^2 C_2. \tag{33}$$

From (25) and (30) one gets for $R \to \infty$ the following result

$$C_2^\kappa = \tfrac{1}{4}\kappa^2 \sinh^2\left(\frac{2P_2}{\kappa}\right) + P_1^2 = \tfrac{1}{8}\kappa^2\left(\cosh\left(\frac{4P_2}{\kappa}\right) - 1\right) + P_1^2. \tag{34}$$

The comultiplication table (22) implies that

$$\begin{aligned}
\Delta(T) &= T \otimes e^{-2P_2/\kappa} + e^{2P_2/\kappa} \otimes T + \frac{2}{\kappa} e^{P_2/\kappa}\{Q_+ \otimes Q_+ - Q_- \otimes Q_-\} e^{-P_2/\kappa} \\
\Delta(P_1) &= P_1 \otimes e^{-2P_2/\kappa} + e^{2P_2/\kappa} \otimes P_1 - \frac{2}{\kappa} e^{P_2/\kappa}(Q_+ \otimes Q_+ + + Q_- \otimes Q_-\} e^{-P_2/\kappa} \\
\Delta(P_2) &= P_2 \otimes 1 + 1 \otimes P_2 \\
\Delta(Q_\pm) &= Q_\pm \otimes e^{-P_2/\kappa} + e^{P_2/\kappa} Q_\pm
\end{aligned} \tag{35}$$

and after contraction the antipodes are

$$S(T) = -T - \frac{1}{\kappa} P_1 \qquad S(Q_\pm) = -Q_\pm \qquad S(P_r) = -P_r \qquad r = 1, 2. \tag{36}$$

(*b*) *Quantum $D = 2$ Poincaré SUSY ($q \neq 1$).* For *all real values of q* the following rescaling

$$L_1 = RP_0 \qquad L_2 = RP_1 \qquad L_3 = L \qquad V_\pm = \sqrt{R}\, q^{\pm 1/2}(q + q^{-1})^{-1} S_\pm \tag{37}$$

provides the finite limit $R \to \infty$ of the quantum superalgebra (18)-(20). We obtain ($\mu = 0, 1$):

$$\begin{aligned}
&\{S_\pm, S_\pm\} = 2(P_1 \mp P_0) \\
&\{S_+, S_-\} = 0 \qquad [P_\mu, P_\nu] = 0 \\
&[L, S_\pm] = \pm\tfrac{1}{2} S_\pm \qquad [P_\mu, Q_\pm] = 0 \\
&[L, P_0] = -P_1 \qquad [L, P_1] = -P_0
\end{aligned} \tag{38}$$

and for the Casimir rescaled according to (33) we get

$$C_2 = P_\mu P^\mu = P_1^2 - P_0^2. \tag{39}$$

We see that the superalgebra structure for the contraction given by (37) describes *classical* $D = 2$ Euclidean superalgebra. The difference however, will appear in the comultiplication rules

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes q^{-4L} + q^{4L} \otimes P_0 + \tfrac{1}{2} q^{2L}\{(q^2 - 1) S_- \otimes S_- + (q^{-2} - 1) S_+ \otimes S_+\} q^{-2L} \\
\Delta(P_1) &= P_1 \otimes q^{-4L} + q^{4L} \otimes P_1 + \tfrac{1}{2} q^{2L}\{(q^{-2} - 1) S_+ \otimes S_+ - (q^2 - 1) S_- \otimes S_-\} q^{-2L} \\
\Delta(L) &= L \otimes 1 + 1 \otimes L \\
\Delta(S_\pm) &= S_\pm \otimes q^{-2L} + q^{2L} \otimes S_\pm
\end{aligned} \tag{40}$$

and in the formulae for the antipode

$$\begin{aligned}
&S(L) = -L \qquad S(S_\pm) = -q^{\mp 1} S_\pm \\
&S(P_0) = -\tfrac{1}{2}(q^2 + q^{-2}) P_0 + \tfrac{1}{2}(q^{-2} - q^2) P_1 \\
&S(P_1) = \tfrac{1}{2}(q^{-2} - q^2) P_0 - \tfrac{1}{2}(q^2 + q^{-2}) P_1.
\end{aligned} \tag{41}$$

We would like to mention that similar structure of the q-deformed $D=2$ supersymmetry was obtained in [13] by studying the q-deformation of SL(1|1) superalgebra with the difference between classical and quantum algebra appearing in the comultiplication rules and the formulae for antipodes.

(*c*) *Quantum $D=2$ Poincaré SUSY ($q=1, \kappa \neq 0$).* The third non-trivial contraction of the quantum superalgebra $U_q(\mathrm{OSp}(1|2))$ is provided by the rescaling

$$L_1 = RP_0 \qquad L_2 = L \qquad L_3 = RP_1 \qquad V_\pm = \sqrt{R}\,\tilde{S}_\pm \tag{42}$$

and the redefinition (31) of the deformation parameter. In the limit $R \to \infty$ one gets

$$\begin{aligned} &\{S_\pm, S_\pm\} = -\tfrac{1}{2}P_0 \qquad \{S_+, S_-\} = -\tfrac{1}{4}\kappa \sinh\frac{2P_1}{\kappa} \\ &[L, S_\pm] = \tfrac{1}{2}\cosh\frac{2P_1}{\kappa} S_\mp \qquad [L, P_0] = \tfrac{1}{4}\kappa \sinh\frac{2P_1}{\kappa} \\ &[L, P_1] = P_0 \qquad [P_\mu, P_\nu] = 0 \qquad [P_\mu, S_\pm] = 0. \end{aligned} \tag{43}$$

The Casimir rescaled in accordance with (4.4) takes the form

$$C_2 = \tfrac{1}{4}\kappa^2 \sinh^2\left(\frac{2P_1}{\kappa}\right) - P_0^2 \tag{44}$$

and can be compared with the one obtained in [4, 5] for the $D=2$ Minkowski quantum plane. The coproduct formulae look as follows

$$\begin{aligned} &\Delta(P_0) = P_0 \otimes \mathrm{e}^{-2P_1/\kappa} + \mathrm{e}^{2P_1/\kappa} \otimes P_0 \\ &\Delta(P_1 = P_1 \otimes 1 + 1 \otimes P_1 \\ &\Delta(L) = L \otimes \mathrm{e}^{-2P_1/\kappa} + \mathrm{e}^{2P_1/\kappa} L - \frac{2}{\kappa}\mathrm{e}^{P_1/\kappa}(S_+ \otimes S_+ + S_- \otimes S_-)\,\mathrm{e}^{P_1/\kappa} \\ &\Delta(S_\pm) = S_\pm \otimes \mathrm{e}^{-P_1/\kappa} + \mathrm{e}^{P_1/\kappa} \otimes S_\pm \end{aligned} \tag{45}$$

and the antipodes are given by

$$S(P_\mu) = -P_\mu \qquad S(P_\pm) = -S_\pm \qquad S(L) = -L - \frac{P_0}{\kappa}. \tag{46}$$

We have shown here three different contraction limits for the real quantum superalgebra $U_q(\mathrm{OSp}(1|2))$ (the fourth, with L_1 unchanged, and $q \neq 1$, is divergent). Similarly, one can perform the contraction limits for other two real forms of the complex quantum superalgebra $U_q(\mathrm{OSp}(1|2))$†.

We have described here different quantum $D=2$ supersymmetry algebras obtained as the contraction limits of the real quantum superalgebra $U_q(\mathrm{OSp}(1|2))$. It should be mentioned that there are also two other contractions, not corresponding to $D=2$ supersymmetry.

(i) Assuming the rescaling (31) and

$$L_k = RP_k \qquad V_\pm = \sqrt{R}\,Q_\pm \tag{47}$$

one obtains in the limit $R \to \infty$ the conventional $D=3$ Minkowski superalgebra relation with modified comultiplication rules.

(ii) Following the derivation of the q-deformed bosonic creation and annihilation operators by considering the limit $j \to \infty$ of the $(2j+1)$-dimensional realizations of

† One of the authors (JL) was informed by P Kulish that some contractions of $U_q(\mathrm{OSp}(1|2))$ were obtained by him and the Firenze group.

$U_q(SU(2))$ [14], one can derive in an analogous way the algebra of q-deformed fermionic oscillators described by the suitable limits of $V_\pm$ [15].

From our formula (*18*) *it is clear that the quantum Lie algebra* $U_q(Sp(2))$ *is* ***not*** *a* quantum subalgebra of $U_q(OSp(1|2))$. This property is due to the prescription (16*b*) for the generators corresponding to the non-simple roots. If we use however the q-oscillator realization of the q-deformed Cartan–Chevaley basis $(e_{\pm\alpha}, h_\alpha)$ of $U_q(OSp(1|2)$ (see e.g. [16, 17]), one can propose *different* definitions of the generators $e_{\pm 2\alpha}$, e.g. obtained by the q-deformation of the oscillator realizations of OSp(1|2). In such a way one obtains a different q-deformation of the Cartan–Weyl basis of OSp(1|2), with q-deformed Sp(2) being the subalgebra of the q-deformed OSp(1|2). Other relations for q-deformed Cartan–Weyl basis $U_q(OSp(1|2))$ have also been obtained by Kulish [6] using the Fadeev–Reshetikhin–Takhtajan formalism of triangular $L^\pm$ operators [2].

The main aim of this letter was to show the variety of possible contractions for a simple rank-one quantum group. Because the R-matrix and dual description of the generators of the algebra of functions for $OSp_q(1|2)$ is also known [6, 7], one can also perform the contractions of the R-matrix and introduce the dual picture of contracted quantum supergroups.

Finally let us mention that our real aim is the q-deformation of the $D=4$ super-Poincaré algebra [18] as the supersymmetric extension of our results for the q-deformed Poincaré algebra [12]. In such a case one considers the contraction of the real quantum superalgebra $U_q(OSP(1|4; R)$.

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